Instructions

- You're welcome to work on these problems individually or in groups.
- If you have any questions, please don't hesitate to raise your hand and call me over! I'll be happy to help.

Complex Numbers

Q1. *[Warm-up.]* Express the following complex numbers in the form $a + bi$, with $a, b \in \mathbb{R}$.

(a) $(4 - 5i) + (1 + 2i)$.
(b) $(3 + 5i)(1 - i)$.
(c) $(2 + 2i)^2$.
(d) i^5.

Q2. *[Trigonometric form.]* Express the following complex numbers in trigonometric form

$$z = r(\cos\theta + i\sin\theta).$$

(a) i.
(b) $-i$.
(c) 3. (Remember: real numbers *are* compelx numbers – they just have zero imaginary part!)
(d) -3.
(e) $1 + i$.
(f) $3 + 4i$.

Q3. *[Polar form.]* Express the complex numbers in **Q2** in polar form

$$z = re^{i\theta}.$$

Q4. *[De Moivre's Theorem.]* Using mathematical induction, prove that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

for all positive integers n.

Note: The result is also true for negative integers (and zero). Can you prove this too?

Q5. *[nth roots of complex numbers.]* Using De Moivre's theorem, find the following roots. Express your answers in both standard and trigonometric forms.

Remember – the strategy is as follows. If we have a complex number, like $w = 4i$, and we want to find its 5th roots say, then we are looking for all complex numbers z that satisfy $z^5 = w$. Here's how De Moivre helps us:

- Write w and z in polar form. For instance, if $w = 4i$, then $w = 4(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$. Since we know nothing about z, we can just write $z = r(\cos\theta + i\sin\theta)$ where r and θ are to be determined.
- Then, by De Moivre's, we can re-write the condition $z^5 = w$ as

$$r^5(\cos 5\theta + i\sin 5\theta) = 4(\cos(\tfrac{\pi}{2}) + i\sin(\tfrac{\pi}{2})).$$

- Now we have to match-up the r's and θ's! We get

$$r^5 = 4 \quad \text{and} \quad 5\theta = \frac{\pi}{2} + 2k\pi \quad (k \in \mathbb{Z}).$$

Remembering that r has to be a *positive real number*, we can find r as $r = \sqrt[4]{2}$. (This is our "usual" fourth root of 2, approximately equal to $1.189207\ldots$). But we get *five* different possibilities for θ:

$$\theta = \frac{\pi}{10} + \frac{2}{5}k\pi \quad (k \in \mathbb{Z}) \implies \theta = \frac{\pi}{10}, \frac{5\pi}{10}, \frac{9\pi}{10}, \frac{13\pi}{10} \text{ or } \frac{17\pi}{10}.$$

All in all, this gives us 5 fifth roots of $4i$: they are of the form

$$z = \sqrt[4]{2}(\cos\theta + i\sin\theta),$$

with θ given as above.

(a) The square roots of i.

(b) The third roots of 8.

(c) The fourth roots of 1. (See also **Q6**.)

Q6. *[Roots of unity.]* The complex solutions to the equation $z^n = 1$ are called the nth roots of unity. There are n distinct nth roots of unity, for each positive integer n. Let

$$\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)..$$

(a) Show that ω is an nth root of unity. [**Hint:** What is ω^n?]

(b) Show that $\omega^2,\ \omega^3,\ \ldots, \omega^{n-1}$ are also nth roots of unity and that no two of them are equal. So, together with ω, these are all nth roots of unity.

(c) Determine

$$\omega + \omega^2 + \cdots + \omega^{n-1} \quad \text{and} \quad \omega\omega^2 \cdots \omega^{n-1}.$$

The next three problems are all related.

Q7. Let $z(t) = \cos 2\pi t + i \sin 2\pi t$, where t is a real number such that $0 \leq t \leq 1$.

(a) Imagining $z(t)$ as describing the position of a complex number in the complex plane at time t. At $t = 0$, the complex number is $z(0) = 1$ and as t increases, $z(t)$ moves in a path that ends back at $z(1) = 1$ at $t = 1$. Plot this path.

(b) Express the complex number $z(t)$ in polar form as $z(t) = r(t)e^{i\theta(t)}$. How does $\theta(t)$ change as t goes from $t = 0$ to $t = 1$?

(c) Looking at part (a), we know that $z(0) = 1$ and $z(1) = 1$, so they should have the same polar form. Does this contradict what you found in part (b)?

Q8. Repeat **Q7** but now let t go from 0 to 2. What do you notice? What if you let t go from 0 to 3?

Q9. Going back to **Q7**(b), pick one of the two square roots of $z(t)$ using your chosen polar form. Let's call it $\sqrt{z(t)}$.

(a) What are $\sqrt{z(0)}$ and $\sqrt{z(1)}$?

(b) Recalling that $z(0) = z(1) = 1$, what do your finding say about the usage of the notation $\sqrt{z}$ for square roots of complex numbers?

Note: **Q9–Q10** are related to a phenomenon known as *monodromy*!